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Geodetic Domination Integrity of Thorny Graphs

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Abstract — The concept of geodetic domination integrity is a crucial parameter when examining the potential damage to a network. It has been observed that the removal of a geodetic set from the network can increase its vulnerability. This study explores the geodetic domination integrity parameter and presents general results on the geodetic domination integrity values of thorn ring graphs, n-sunlet graphs, thorn path graphs, thorn rod graphs, thorn star graphs, helm graphs, E_p^t tree graphs, dendrimer graphs, spider graphs, and bispider graphs, which are the frequently used graph classes in the literature.

Keywords *Geodetic domination integrity, geodetic dominating set, geodetic set, dominating set*

Mathematics Subject Classification (2020) 05C69, 05C76

1. Introduction

Measuring the stability and reliability of communication networks is critical in today's rapidly growing and changing communication infrastructures. The centers of a network can be modeled as the vertices and the connecting lines as the edges of the graph. An important question is how long the network's communication will last if some vertices or edges of the graph modeling a network are damaged. The measurement of the resilience of a network after the damage of some centers or connection lines in a communication network, until the communication is lost in the remaining network, is called vulnerability [1]. Various parameters have been defined to measure the vulnerability of networks. Some of these measurements are connectivity, integrity, domination integrity, and toughness [2]. Different versions of these parameters have been defined according to the features needed in the networks [2–4]. The geodetic domination integrity is one of the newly defined parameters [5]. Finding a geodesic path in any network to optimize time and cost plays an important role.

A geodetic path is the shortest path between two vertices. The combination of the shortest paths between the elements of the geodetic set of the graph modeling, the network covers the entire network. Transportation networks are required to pass through critical centers and to minimize the cost of logistics expenses. The analysis of this set plays an important role in optimizing traffic flow, planning public transport networks, and improving road safety. Geodetic set analysis helps to find solutions to problems, such as identifying areas with traffic congestion or determining alternative transportation routes. Damage to the geodetic dominating set can disrupt all communication in the network. The geodetic domination integrity is an important parameter to investigate the network-wide damage

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because removing a geodetic set from the network increases its vulnerability [6]. Therefore, geodetic domination integrity has a wide research area in graph theory, which motivated us to study geodetic domination integrity of graphs.

In this study, the geodetic domination integrity parameter is studied, and general results are obtained and proved for thorn graphs, dendrimer graphs, helm graphs, E_p^t tree graphs, and spider graphs, frequently used graph classes in the literature.

2. Preliminaries

This section provides some basic notions to be required for the following sections. Throughout the paper, simple graphs are considered, and the books [1, 7–10] are used for the basic definitions. For any graph $G = (V, E)$, the order is $n = |V(G)|$, and the size is $m = |E(G)|$. The set $N(v) = \{u_i : d(v, u_i) = 1, u_i \in V(G)\}$ is the open neighborhood of $v \in V(G)$, and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is defined by $d(v) = |N(v)|$. For $X \subseteq V(G)$, let $G[X]$ be the subgraph of graph G induced by X, $N(X) = \sum_{x\in X} N(x)$, and $N[X] = \sum_{x\in X} N[x]$. The length of a shortest path $x - y$ in a connected graph G is the distance between x and y, denoted by $d(x, y)$, and the diameter of a graph is defined by $\text{diam}(G) = \max_{x,y\in V(G)}\{d(x, y)\}$. An $x - y$ geodesic is a path of length $d(x, y)$, and the closed interval $I[x, y]$ consists of x, y, and all the vertices contained on some geodesic $x - y$ where $I[S] = \bigcup_{x,y\in S} I[x, y]$, for $S \subseteq V(G)$. If $I[S] = V(G)$, then S is a geodetic set. The minimum cardinality of a geodetic set is the geodetic number of G, denoted by $g(G)$. A geodetic set is called a $g(G)$-set if its cardinality is $g(G)$ [6]. $S \subseteq V(G)$ is a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S; in other words, if every vertex of G is dominated by some vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by $\gamma(G)$ [1]. If a dominating set is also a geodetic set, then the set is called a geodetic dominating set, and the minimum cardinality of a geodetic dominating set in G is called the geodetic domination number, denoted by $\gamma_g(G)$ [11].

A communication network consists of centers and connection lines that enable these centers to communicate with each other. The graph's resistance following the breakdown of certain centers or connections is called vulnerability in a communication network. There are some parameters to measure vulnerability [2]. One of which, the geodetic domination integrity DI_g, was defined in 2021 by Balaraman et al. [5].

Definition 2.1. [5] The geodetic domination integrity of a graph G is defined by

$$DI_g(G) = \min_{S\subseteq V(G)}\{|S| + m(G - S)\}$$

where the order of the greatest component in $G - S$ is indicated by $m(G - S)$, and S is a geodetic dominating set of G. If $DI_g(G) = |S| + m(G - S)$, then a subset S of $V(G)$ is a DI_g-set.

Lemma 2.2. [5] The general results for the geodetic domination integrity of some known graphs are as follows:

i. $DI_g(K_n) = n$

ii. $DI_g(K_{1,n-1}) = n$

iii. $DI_g(K_{r,s}) = \min\{r, s\} + 1$, for $r, s \geq 2$

iv. $DI_g(W_n) = \left\lceil \frac{n-1}{2} \right\rceil + 2$, for $n \geq 5$

v. $DI_g(C_n) = \lceil \frac{n}{3} \rceil + 2, n \geq 6$

vi. $DI_g(P_n) = \left\lceil \frac{n+2}{3} \right\rceil + 2$

vii. $DI_g(G) = 6$ where G is the Petersen graph.

3. Geodetic Domination Integrity

In this section, the geodetic domination integrity values of thorn ring graphs, n-sunlet graphs, thorn path graphs, thorn rod graphs, thorn star graphs, helm graphs, E_p^t tree graphs, dendrimer graphs, regular dendrimer graphs, spider graphs, and bispider graphs, were analyzed, and general formulas were obtained based on the order of the graphs. Across this study, let $I_n := \{1, 2, 3, \cdots, n\}$.

Definition 3.1. [12] Let k be a non-negative integer. A thorn ring graph, denoted by $C_{n,k}$, is constructed by adding k pendant edges to each vertex of the cycle graph C_n.

Figure 1 illustrates the thorn ring $C_{8,3}$.

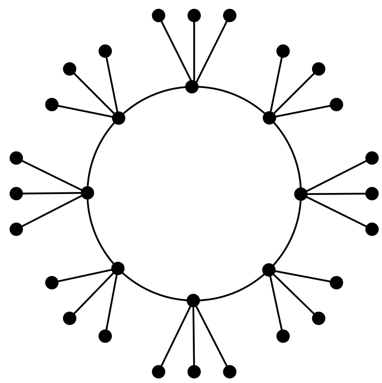

Figure 1. $C_{8,3}$ thorn ring

Theorem 3.2. Let $C_{n,k}$ be a thorn ring graph. Then,

$$DI_g(C_{n,k}) = nk + \lceil 2\sqrt{n} \rceil - 1$$

Proof. Let $C_{n,k}$ be a thorn ring graph where $\{x_i : \deg(x_i) = 1, i \in I_{nk}\}$ are pendant vertices, S be the geodetic dominating set, and $m(C_{n,k} - S)$ be the largest component in $C_{n,k} - S$. Let $X \subseteq V(C_{n,k})$ with $|X| = r$ such that $S = \{x_1, x_2, \cdots, x_{nk}\} \cup X$ is a geodetic dominating set. Then, $|S| = nk + r$ and the number of components in $C_{n,k} - S$, denoted by $\omega(C_{n,k} - S)$, is at most r. Hence, $m(C_{n,k} - S) \geq \dfrac{n + nk - (nk + r)}{r}$ which implies

$$DI_g(C_{n,k}) \geq \min_r \left\{ nk + r + \frac{n - r}{r} \right\}$$

For $r \geq 0$, the minimum integer value of the function $f(r) = nk + r + \frac{n-r}{r}$ is $nk + \lceil 2\sqrt{n} \rceil - 1$. Since geodetic domination integrity is an integer value,

$$DI_g(C_{n,k}) = nk + \lceil 2\sqrt{n} \rceil - 1$$

□

Definition 3.3. [13] An n-sunlet graph is obtained from the cycle graph C_n by adding n pendant edges to each vertex of G and is denoted by S_n.

Figure 2 illustrates the 8-sunlet graph S_8.

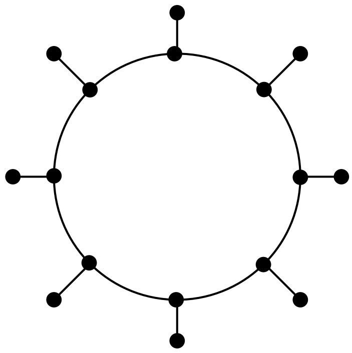

Figure 2. 8-sunlet graph

Theorem 3.4. Let S_n be an n-sunlet graph. Then,

$$DI_g(S_n) = n + \lceil 2\sqrt{n} \rceil - 1$$

PROOF. Since $C_{n,1} \cong S_n$, it follows from Theorem 3.2 that $DI_g(S_n) = n + \lceil 2\sqrt{n} \rceil - 1$. □

Definition 3.5. [14] Let p and u be non-negative integers. A thorn path graph, denoted by $P_{n,p,u}$, is constructed by adding u pendant edges to both the initial and the terminal vertices of the path graph P_n, and p pendant edges to each internal vertex.

Figure 3 illustrates the thorn path $P_{6,2,3}$.

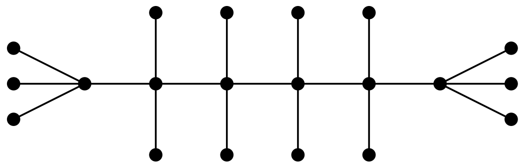

Figure 3. $P_{6,2,3}$ thorn path

Theorem 3.6. Let $P_{n,p,u}$ be a thorn path graph. Then,

$$DI_g(P_{n,p,u}) = (n-2)p + 2u + \lceil 2\sqrt{n+1} \rceil - 2$$

PROOF. Let $P_{n,p,u}$ be a thorn path graph with pendant vertices $\{x_k : \deg(x_k) = 1, k \in I_{(n-2)p}\}$ and $\{y_j : \deg(y_j) = 1, j \in I_{2u}\}$, S be a geodetic dominating set, and $m(P_{n,p,u} - S)$ be the largest component in $P_{n,p,u} - S$. Let $X \subseteq V(P_{n,p,u})$ and $|X| = r$ such that $S = \{x_1, x_2, \cdots, x_{(n-2)p}\} \cup \{y_1, y_2, \cdots, y_{2u}\} \cup X$ be a geodetic dominating set removed. Hence, $|S| = (n-2)p + 2u + r$ and the number of components in $P_{n,p,u} - S$ is $\omega(P_{n,p,u} - S) \leq r + 1$, implying

$$m(P_{n,p,u} - S) \geq \frac{n + (n-2)p + 2u - ((n-2)p + 2u + r)}{r+1}$$

Therefore,

$$DI_g(P_{n,p,u}) \geq \min_r \left\{ (n-2)p + 2ur + \frac{n-r}{r+1} \right\}$$

For $r \geq 0$, $f(r) = (n-2)p + 2u + r + \frac{n-r}{r+1}$ is the function, and its lowest value is $(n-2)p + 2u + \lceil 2\sqrt{n+1} \rceil - 2$. Since the geodetic domination integrity is an integer value,

$$DI_g(P_{n,p,u}) = (n-2)p + 2u + \lceil 2\sqrt{n+1} \rceil - 2$$

□

Definition 3.7. [12] Let n and k be non-negative integers. A thorn rod graph, denoted by $P_{n,k}$, is constructed by adding k pendant edges to each of the initial and the terminal vertices of the path graph P_n.

Figure 4 illustrates the thorn rod $P_{6,3}$.

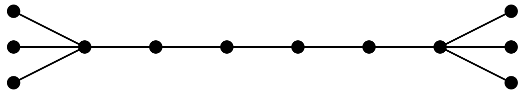

Figure 4. $P_{6,3}$ thorn rod

Theorem 3.8. Let $P_{n,k}$ be a thorn rod graph. Then,

$$DI_g\left(P_{n,k}\right) = 2k + \lceil 2\sqrt{n+1} \rceil - 2$$

PROOF. Since $P_{n,k} \cong P_{n,0,k}$, as a result of Theorem 3.6 that $DI_g\left(P_{n,k}\right) = 2k + \lceil 2\sqrt{n+1} \rceil - 2$. □

Definition 3.9. [14] Let p and k be non-negative integers. A thorn star graph, denoted by $S_{n,p,k}$, is constructed by attaching k pendant edges to each pendant vertex and p pendant edges to the central vertex of the complete bipartite graph $K_{1,n}$.

Figure 5 illustrates the thorn star $S_{4,2,3}$.

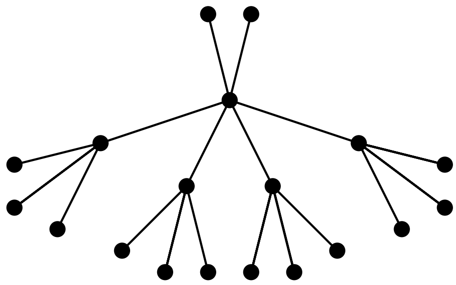

Figure 5. $S_{4,2,3}$ thorn star

Theorem 3.10. Let $S_{n,p,k}$ be a thorn star graph. Then,

$$DI_g\left(S_{n,p,k}\right) = 2 + p + nk$$

PROOF. Assume that $S_{n,p,k}$ is a thorn star graph, v is its central vertex, $\{x_i : \deg(x_i) = 1, i \in I_{nk}\}$, and $\{y_j : \deg(y_j) = 1, j \in I_p\}$ are pendant vertices. There is an equal number of pendant vertices and elements in the geodetic dominating set S, i.e., $|S| = p + nk$. Thus, $\gamma_g\left(S_{n,p,k}\right) = p + nk$. Removing the set $\{x_1, x_2, \cdots, x_{nk}\} \cup \{y_1, y_2, \cdots, y_p\}$ from the graph leaves a $K_{1,n}$ star graph with $n+1$ vertices. Removing the central vertex v from $K_{1,n}$ leaves n isolated vertices. Therefore, if the dominating set $S = \{x_1, x_2, \cdots, x_{nk}\} \cup \{y_1, y_2, \cdots, y_p\} \cup \{v\}$ is removed from $S_{n,p,k}$, then the largest component is $m\left(S_{n,p,k} - S\right) = 1$. Thus,

$$\begin{aligned} DI_g\left(S_{n,p,k}\right) &= |S| + m\left(S_{n,p,k} - S\right) \\ &= 1 + nk + p + 1 \\ &= 2 + p + nk \end{aligned}$$

The geodetic domination integrity value of the thorn star graph is

$$DI_g\left(S_{n,p,k}\right) = 2 + p + nk$$

□

Definition 3.11. [15] A Helm graph is constructed by adding a pendant edge to every vertex of the wheel graph W_n with the exception of the central vertex and denoted by H_n. H_n has $2n+1$ vertices and $3n$ edges.

Figure 6 illustrates the Helm graph H_6.

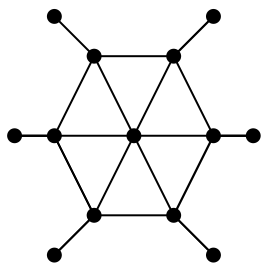

Figure 6. H_6 graph

Theorem 3.12. Let H_n be a Helm graph. Then,

$$DI_g\left(H_n\right) = n + \lceil 2\sqrt{n} \rceil$$

Proof. Let H_n be a Helm graph with pendant vertices $\{x_i : \deg\left(x_i\right) = 1, i \in I_n\}$, v be the central vertex, S be a geodetic dominating set, and $m\left(H_n - S\right)$ be the largest component in $H_n - S$. The Helm graph H_n contains a cycle graph C_n with n vertices, and each vertex of C_n is adjacent to the central vertex v. Let $X \subseteq V\left(H_n\right)$ with $|X| = r$, and consider $S = \{v\} \cup \{x_1, x_2, \cdots, x_n\} \cup X$ as the dominating set. Then, $|S| = n + 1 + r$ and $\omega\left(H_n - S\right) \leq r$, implying

$$m\left(H_n - S\right) \geq \frac{2n + 1 - (n + 1 + r)}{r}$$

Hence,

$$DI_g\left(H_n\right) \geq \min_r \left\{ n + 1 + r + \frac{n - r}{r} \right\}$$

For $r \geq 0$, the function $f(r) = n + 1 + r + \frac{n-r}{r}$ has the minimum value $n + 2\sqrt{n}$. Since the geodetic domination integrity is an integer value,

$$DI_g\left(H_n\right) = n + \lceil 2\sqrt{n} \rceil$$

□

Definition 3.13. [10] The graph E_p^t is a graph with t legs, each containing p vertices.

Figure 7 illustrates the graph E_3^5.

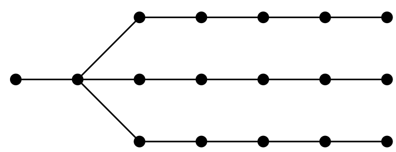

Figure 7. E_3^5 graph

Theorem 3.14. Let E_p^t be a tree graph. Then, the geodetic domination integrity of E_p^t is given by

$$DI_g\left(E_p^t\right) = \begin{cases} t\lfloor \frac{p+2}{3} \rfloor + 3, & n \equiv 1 \pmod 3 \\ t\lfloor \frac{p+2}{3} \rfloor + 4, & \text{otherwise} \end{cases}$$

Proof. Let ${E_p}^t$ be a tree graph where x represents the vertex with degree 1, y represents the vertex with the maximum degree, and $u_1, u_2, \cdots, u_p$ represent the vertices on the paths (for every path t). Let S be a geodetic dominating set, $m\left({E_p}^t - S\right)$ be the largest component in ${E_p}^t - S$, and $I_G[S]$ denote the union of all the geodetic sets $I_G[a, b]$, for all $a, b \in S$.

i. For $n \equiv 0 \pmod 3$, let $S = \{u_{3k} : 1 \leq k \leq \frac{p}{3}\} \cup \{x\} \cup \{y\}$. Then, $|S| = t\left\lfloor \frac{p+2}{3} \right\rfloor + 2$. Since $u_{3k-2}, u_{3k-1} \in N\left(u_{3k}\right)$ and $I_G[S] = V\left(E_p^t\right)$, S is a geodetic dominating set for E_p^t. Removing S from the graph yields $m\left(E_p^t - S\right) = 2$. Hence, $DI_g\left(E_p^t\right) = t\left\lfloor \frac{p+2}{3} \right\rfloor + 4$.

ii. For $n \equiv 1 \pmod 3$, let $S = \left\{u_{3k+1} : 0 \leq k \leq \frac{p-1}{3}\right\} \cup \{x\}$. Then, $|S| = t\left\lfloor \frac{p+2}{3} \right\rfloor + 1$. Since $u_{3k}, u_{3k+2} \in N\left(u_{3k+1}\right)$ and $I_G[S] = V\left(E_p^t\right)$, S is a geodetic dominating set for E_p^t. Removing S from the graph yields $m\left(E_p^t - S\right) = 2$. Hence, $DI_g\left(E_p^t\right) = t\left\lfloor \frac{p+2}{3} \right\rfloor + 3$.

iii. For $n \equiv 2 \pmod 3$, let $S = \left\{u_{3k+2} : 0 \leq k \leq \frac{p-2}{3}\right\} \cup \{x\} \cup \{y\}$. Then, $|S| = t\left\lfloor \frac{p+2}{3} \right\rfloor + 2$. Since $u_{3k}, u_{3k+1} \in N\left(u_{3k+2}\right)$ and $I_G[S] = V\left(E_p^t\right)$, S is a geodetic dominating set for E_p^t. Removing S from the graph yields $m\left(E_p^t - S\right) = 2$. Hence, $DI_g\left(E_p^t\right) = t\left\lfloor \frac{p+2}{3} \right\rfloor + 4$.

From *i-iii*,

$$DI_g\left(E_p^t\right) = \begin{cases} t\lfloor \frac{p+2}{3} \rfloor + 3, & n \equiv 1 \pmod 3 \\ t\lfloor \frac{p+2}{3} \rfloor + 4, & \text{otherwise} \end{cases}$$

□

Definition 3.15. [16] Let k and n be positive integers. Dendrimer graphs $D_{k,n}$ are constructed by adding k degree-one vertices to the vertices with degree one in the graph $D_{k,0}$, initially defined as D_0, for a total of n repetitions.

Figure 8 illustrates the graph D_0, while Figure 9 shows the graphs $D_{2,1}$ and $D_{2,2}$.

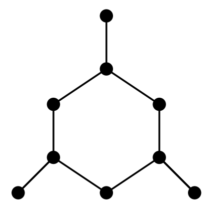

Figure 8. D_0 dendrimer

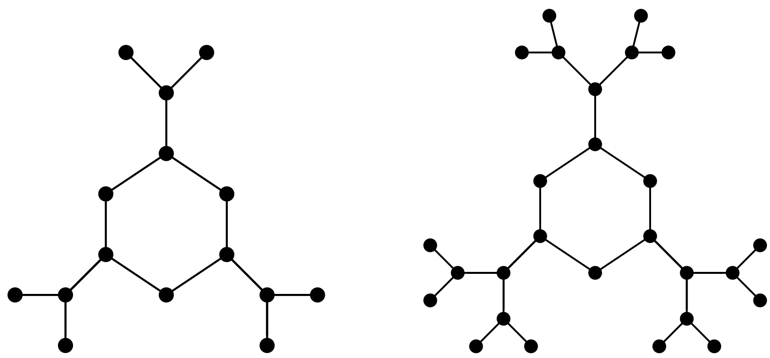

Figure 9. $D_{2,1}$ and $D_{2,2}$ dendrimer

Theorem 3.16. [4] Let H_n^k be a complete k-ary tree of height $n-1$. Then, the domination number is given by

$$\gamma\left(H_n^k\right) = \begin{cases} \frac{k\left(k^{(n/3)}-1\right)}{7}, & \text{if } n \equiv 0 \pmod 3 \\ 1 + \frac{k^2\left(k^{\left(\frac{n-1}{3}\right)}-1\right)}{7} & \text{if } n \equiv 1 \pmod 3 \\ 1 + \frac{k^3\left(k^{\left(\frac{n-2}{3}\right)}-1\right)}{7} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Theorem 3.17. Let $D_{k,n}$ be a dendrimer graph. For $k, n > 2$, the geodetic domination integrity of $D_{k,n}$ is

$$DI_g\left(D_{k,n}\right) = 3\left(\gamma\left(H_{n-2}^k\right) + k^n\right) + k + 4$$

PROOF. Let $D_{k,n}$ be a dendrimer graph with pendant vertices $\{x_i : \deg(x_i) = k^n, i \in I_{3(k^n)}\}$, S be a geodetic dominating set, and $m(D_{k,n} - S)$ be the largest component in $D_{k,n} - S$. The number of pendant vertices is equal to the number of vertices composing the smallest geodetic set of $D_{k,n}$. Therefore, $g(D_{k,n}) = 3k^n$. These vertices are the elements of the smallest geodetic dominating set S. They have the property of dominating the vertices that are at distance of n edges from the center, forming a C_6 graph. Thus, to find the dominating set of $D_{k,n}$, it suffices to find the minimum dominating set of $D_{k,n-2}$. The graph $D_{k,n-2}$ is formed by attaching three different H^k_{n-2} k-ary trees to the three vertices of a C_6 graph such that it is regular. Therefore, $|S| = 3\left(\gamma\left(H^k_{n-2}\right) + k^n\right) + 3$. In this case, $m(D_{k,n} - S) = k + 1$. Hence,

$$|S| + m(D_{k,n} - S) \geq 3\left(\gamma\left(H^k_{n-2}\right) + k^n\right) + 3 + (k+1)$$

which leads to

$$DI_g(D_{k,n}) = 3\left(\gamma\left(H^k_{n-2}\right) + k^n\right) + k + 4$$

□

Definition 3.18. [16] A regular dendrimer graph is a tree consisting of a central vertex v and each non-pendant vertex has a degree d two or more. In regular dendrimers, the distance from the central vertex to each pendant vertex is called the radius and is denoted by k. Regular dendrimer graphs are denoted by $T_{k,d}$.

Figure 10 illustrates the regular dendrimers $T_{2,4}$ and $T_{3,4}$.

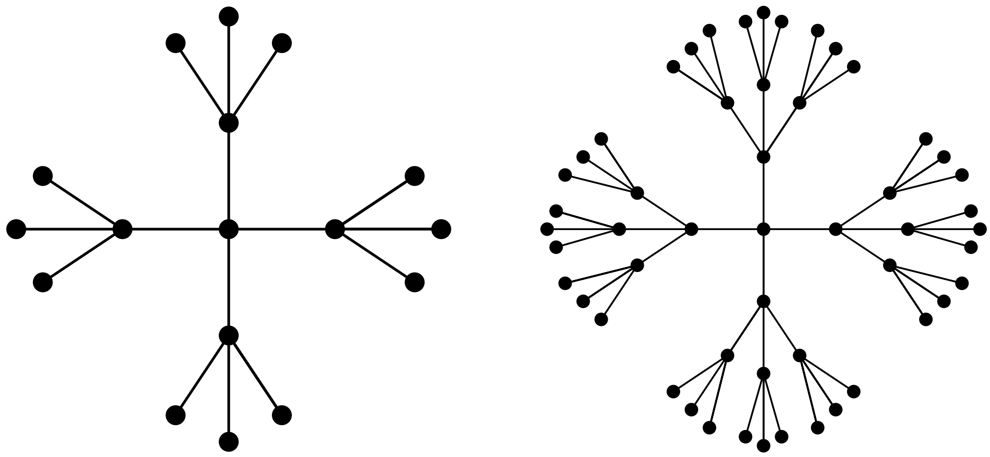

Figure 10. Regular dendrimers $T_{2,4}$ and $T_{3,4}$

Theorem 3.19. [17] Let $T_{k,d}$ be a regular dendrimer graph. Then, the domination number is given by

$$\gamma(T_{k,d}) = \begin{cases} 1 + \frac{(d-1)^k - d + 1}{d-2}, & k \text{ is odd} \\ \frac{(d-1)^k - 1}{d-2}, & k \text{ is even} \end{cases}$$

Theorem 3.20. For $k, d > 2$, the geodetic domination integrity of a regular dendrimer graph is

$$DI_g(T_{k,d}) = d + \gamma(T_{k-2,d}) + d(d-1)^{k-1}$$

PROOF. Suppose that $T_{k,d}$ is a regular dendrimer graph where $k, d > 2$, v is the central vertex, $\{x_i : d(v, x_i) = k, i \in I_{d(d-1)^{k-1}}\}$ are the pendant vertices, S is a geodetic dominating set, and $m(T_{k,d} - S)$ is the largest component in $T_{k,d} - S$. The number of vertices in $T_{k,d}$ forming the minimum geodetic set is equal to the number of pendant vertices. Hence, $g(T_{k,d}) = d(d-1)^{k-1}$. These vertices form the minimum geodetic dominating set S, and they dominate the vertices at distance $k-1$ from the central vertex. Therefore, to find the geodetic dominating set of $T_{k,d}$, it is sufficient to find the minimum dominating set of $T_{k-2,d}$. Hence, $|S| = \gamma\left(T_{k-2,d}\right)+d(d-1)^{k-1}$. In this case, $m\left(T_{k,d}-S\right)=d$. Therefore,

$$|S| + m\left(T_{k,d}-S\right) \geq \gamma\left(T_{k-2,d}\right) + d(d-1)^{k-1} + d$$

which leads to

$$DI_g\left(T_{k,d}\right) = d + \gamma\left(T_{k-2,d}\right) + d(d-1)^{k-1}$$

□

Definition 3.21. [18] A spider graph is constructed by adding a pendant edge to each pendant vertex of the $K_{1,k}$ graph and is denoted by S_k^*.

Figure 11 shows the spider graph S_3^*.

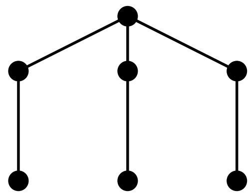

Figure 11. S_3^* graph

Theorem 3.22. Let S_k^* be a spider graph. Then, the geodetic domination integrity of S_k^* is

$$DI_g\left(S_k^*\right) = k+2$$

Proof. Assume that S_k^* is a spider graph with a central vertex v, $\{x_i : \deg(x_i) = 1, i \in I_k\}$ are pendant vertices, and S is a geodetic dominating set. The number of vertices forming the minimum dominating set of S_k^* is equal to the number of pendant vertices. Hence, $g\left(S_k^*\right) = k$. Removing the set $\{x_1, x_2, \cdots, x_k\}$ from the graph leaves a star graph $K_{1,k}$ with $k+1$ vertices. Removing the central vertex v from $K_{1,k}$ leaves k isolated vertices. Therefore, if $S = \{x_1, x_2, \cdots, x_k\} \cup \{v\}$ is removed from S_k^*, then the largest component is $m\left(S_k^* - S\right) = 1$. Thus,

$$\begin{aligned} DI_g\left(S_k^*\right) &= |S| + m\left(S_k^* - S\right) \\ &= k+1+1 \\ &= k+2 \end{aligned}$$

Therefore, the geodetic domination integrity of the spider graph is

$$DI_g\left(S_k^*\right) = k+2$$

□

Definition 3.23. [18] A bispider graph is constructed by adding one edge between the central vertices of two S_k^* graphs and is denoted by $S_{r,s}^*$.

Figure 12 shows the bispider graph $S_{3,3}^*$.

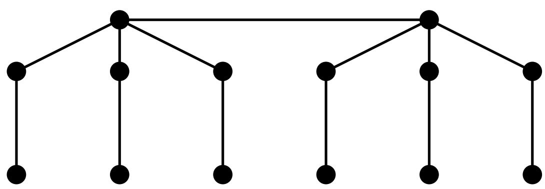

Figure 12. $S_{3,3}^*$ graph

Theorem 3.24. For a bispider graph $S^*_{r,s}$, the geodetic domination integrity is

$$DI_g\left(S^*_{r,s}\right) = r + s + 3$$

PROOF. Suppose that $S^*_{r,s}$ is a bispider graph with central vertices u and v, pendant vertices are $\{x_i : \deg(x_i) = 1, i \in I_r\}$ and $\{y_i : \deg(y_i) = 1, i \in I_s\}$, and S is a geodetic dominating set. The number of vertices forming the minimum geodetic set of $S^*_{r,s}$ is equal to the number of pendant vertices. Hence, $g\left(S^*_{r,s}\right) = r + s$. Removing the set $\{x_1, x_2, \cdots, x_r\} \cup \{y_1, y_2, \cdots, y_s\}$ from the graph leaves two star graphs connected to each other, each with $s+r+2$ vertices. Removing the central vertices u and v from this graph leaves $r+s$ isolated vertices. Therefore, if $S = \{x_1, x_2, \cdots, x_r\} \cup \{y_1, y_2, \cdots, y_s\} \cup \{u\} \cup \{v\}$ is removed from $S^*_{r,s}$, then the largest component is $m\left(S^*_{r,s} - S\right) = 1$. Thus,

$$\begin{aligned} DI_g\left(S^*_{r,s}\right) &= |S| + m\left(S^*_{r,s} - S\right) \\ &= r + s + 2 + 1 \\ &= r + s + 3 \end{aligned}$$

Then, the geodetic domination integrity of the bispider graph is

$$DI_g\left(S^*_{r,s}\right) = r + s + 3$$

□

4. Conclusion

In this study, a newly defined parameter geodetic domination integrity of some classes of graphs, thorn graphs, dendrimer graphs, helm graphs, E^t_p trees, spider graphs, and bispider graphs are investigated, and general formulas are obtained based on the order of the graphs. In future studies, it is recommended to apply this parameter on different types of graphs, especially on transformation graphs of thorny graphs. When a graph is transformed, it loses its old form, and a new graph structure is formed. If it is possible to decode the given graph from the encoded graph in polynomial time, such an operation can be used to analyze various structural properties of the original graph in terms of transformation graphs. As the geodetic domination integrity has a broad research area in graph theory and there is limited work on this newly defined parameter, it is expected that obtaining general results by applying this new parameter to transformation graphs will make a significant contribution to the literature.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's master's thesis supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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